

Optimal Index Reconstitution Strategies

DE Shaw & Co

Abstract

Changes in benchmark composition can have considerable impact on tracking error and will generally entail significant portfolio turnover if a manager wishes to maintain a given tracking error target. Russell Investments, for example, announces its index reconstitutions approximately one month before they take effect. Under these circumstances, it is possible to calculate optimal single-stock strategies for transitioning stocks into and out of a portfolio that tracks one of the Russell indexes.

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1. Introduction

Managing an equity portfolio against a benchmark is, of course, one of the most common forms of investment activity. Quantitative managers, as well as some fundamental managers, base their portfolio construction process on the optimization of information ratios, or the ratio of benchmark-relative return to tracking error, which is the standard deviation of the difference between portfolio and benchmark returns.

Managing a portfolio relative to a benchmark presents interesting challenges when the benchmark itself changes. Benchmarks can change for a variety of reasons: the methodology used to measure the market capitalization of companies may be altered, a constituent company in the index may be delisted, or, most frequently, the underlying index may be reconstituted. Reconstitution events may or may not be preannounced, depending on the index provider. It is standard practice, however, for an index provider to change a given benchmark's composition at the end of the trading day. The family of indexes made available by Russell Investments is based on perhaps the most straightforward methodology in the industry. The Russell indexes are constructed on the basis of market capitalization and adjusted once a year towards the end of June. In the case of the Russell 3000[®] Index¹ ("R3000"), the largest 3,000 companies based on market capitalization at the end of each May are designated as that index's new constituents as of late June. In the small-to-mid cap space, the Russell 2500[™] Index ("R2500") comprises the stocks ranked from 501 to 3,000 in the R3000.

2. Index Reconstitutions and Manager Trade-offs

A change in a given benchmark's composition leads to a trade-off for investment strategies that are managed relative to that benchmark. Other things being equal, tracking error will change on the date of reconstitution because the change in the benchmark alters benchmark-relative risk exposures. Typically, tracking error will increase, and, for passive managers running at extremely low levels of risk (say 10 basis points), this increase generally will correspond to the scale of the reconstitution. If an investment manager wishes to maintain the tracking error of a portfolio at a constantly low level, rebalancing will need to be closely synchronized with the reconstitution. However, the costs of such rebalancing can be high. For one thing, trading in large amounts over narrow windows is akin to salmon swimming upstream in spawning season: predators will arrange their schedules accordingly, so watchful bears on nearby rocks can be expected. Second, trading over short intervals at the same time as other managers with the same incentives that are executing similar trades is likely to result in higher transaction costs than trading at a slower rate. The market-impact component of such costs will generally be higher than average given the volume of trades put on by similarly minded market participants.

Since market impact will vary with the targeted tracking error of a given strategy, it's reasonable to expect that index reconstitutions will affect passive strategies the most and higher risk strategies (targeting 500 or more basis points of tracking error) the least. For large pools of capital invested in "enhanced" strategies, tracking errors will typically range between 50 and 200 basis points. At such risk levels, benchmark changes may adversely affect performance if not managed optimally because the proportional impact of an index reconstitution on managers targeting, say, 100 basis points of tracking error will be substantially greater than the impact on those targeting 500 basis points of risk. In summary, the lower the maximum tracking error that a manager considers

acceptable, the greater the exposure to anticipatory index trading and the higher the costs that are likely to be incurred in the transition from the old benchmark to the new.

Despite these tradeoffs, passive and enhanced managers customarily attempt to mimic the reconstitution timetable in a precise manner. In particular, the benchmark against which risk is calculated switches discretely from old to new over the reconstitution date. It is interesting then to consider the circumstances under which such discrete switching is optimal.

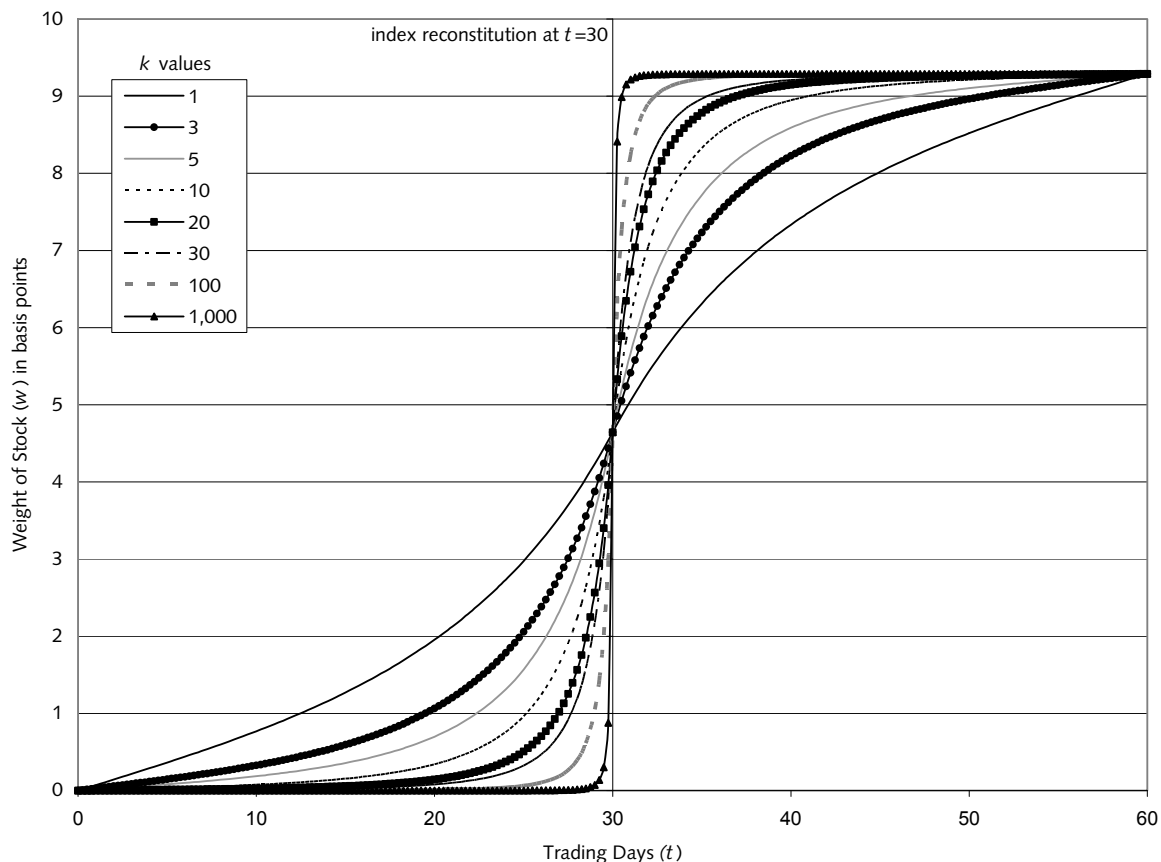
The results of such an exercise strongly suggest that for many institutional investors, current approaches to the management of such policies may be far from optimal. Intriguingly, the performance improvements that might be obtained from a better structured approach rival those typically obtained from so-called enhanced index portfolios, suggesting that smarter indexing may be an untapped and relatively inexpensive source of alpha.

In order to measure this trade-off, the first task is to determine the optimal rebalancing policy for a given degree of risk aversion. With this policy in hand, the trade-off between tracking error and transaction costs can be calculated, and the optimal portfolio path can be determined for given levels of risk aversion. To illustrate the implications of this approach, the results of using this approach on passive R2500 portfolios are shown in the following section. Because the algebraic derivation of the model used to generate these results is both tedious and complicated, the relevant math has been safely cordoned off in the Appendix, where the quantitatively inclined may peruse it at their leisure.

3. Solution Properties

The model in the Appendix derives the optimal solution path for weighting a stock in an index-tracking portfolio in advance of that stock's addition to the benchmark. This assumes that the list of stocks to be added and deleted from the index is known in advance of the reconstitution, as is currently the case with reconstitutions of the Russell indexes. This solution depends, *inter alia*, on trading volume, volatility, and slippage costs. With assumed values for those elements in place (as shown in the Appendix), and taking median parameter values for additions to the R2500 in 2001, Figure 1 shows the optimal time path for a representative addition to the index for varying levels of risk aversion (k) where $w(t)$ represents the weight of the stock in the portfolio at time t . The optimal path of a representative addition to the R2500 is shown for the month before and after the date of the reconstitution ($t = 30$).

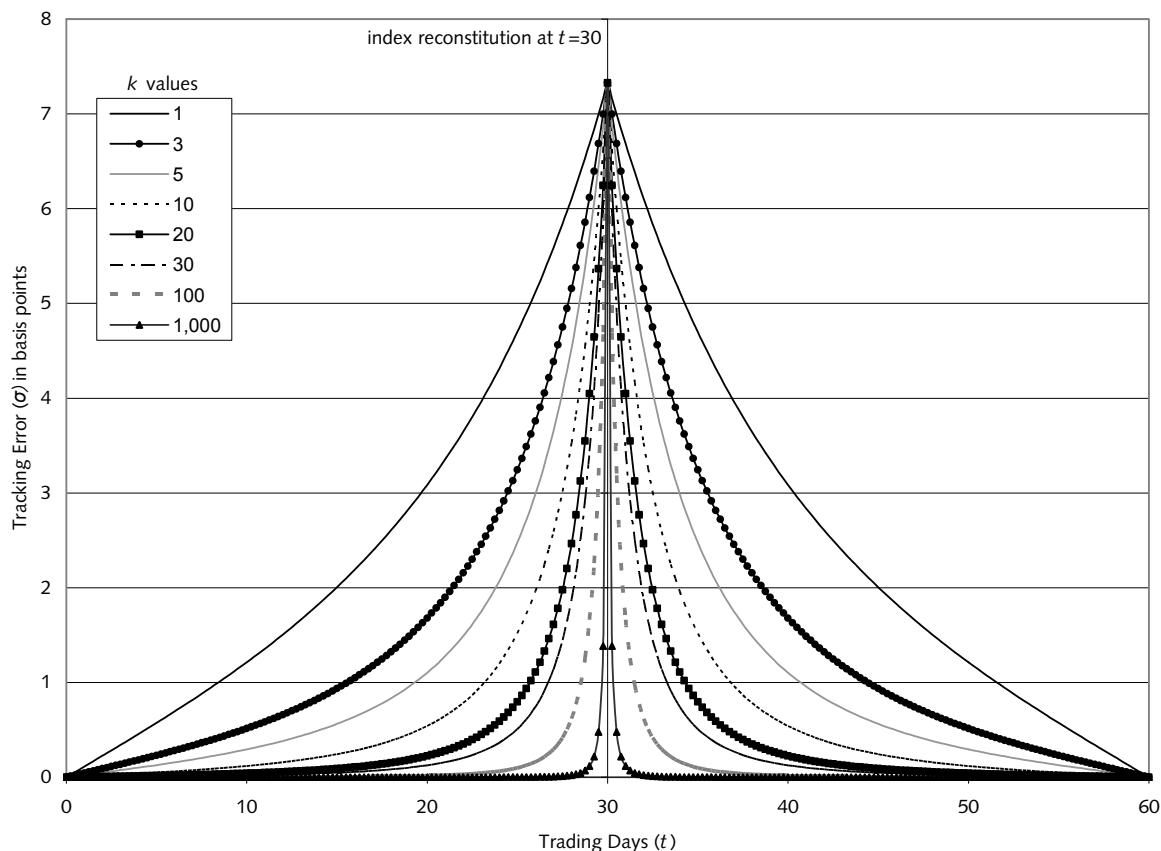
Figure 1: Optimal Time Path for a Representative Addition to the R2500 for Various k Values



These results are in accord with intuition. As $k \rightarrow 0$, risk aversion falls and slippage becomes more important. At the limit, when risk aversion is zero, the optimal path is a straight line. This makes sense because such a policy minimizes slippage and constitutes the optimal policy choice if tracking error is irrelevant. By contrast as $k \rightarrow \infty$, the solution tends to the other extreme. Here w is zero right up until the instant t_r ($t = 30$), at which point it jumps to the new benchmark weight. In this case, any amount of slippage will be tolerated to eliminate tracking error. For levels of risk aversion between the two extremes, the optimal paths follow symmetrical curves around the reconstitution date. In such cases, the strategy begins trading into the new positions ahead of the reconstitution date and continues the adjustment phase after the date of the reconstitution.

Tracking error will likewise vary for different levels of risk aversion, and of course the composition of the benchmark against which tracking error is measured will change on the day of the reconstitution. Figure 2 illustrates the contribution to tracking error (σ) for the same representative R2500 index addition and under the same set of risk aversion values. As risk aversion increases, the tolerance for tracking error decreases and thus less cumulative benchmark-relative exposure is taken over the reconstitution window.

Figure 2: Optimal *Ex-ante* Time Path for the Tracking Error of a Representative Addition to the R2500 for Various *k* Values



This analysis generates a set of “smoothed” paths, each of which is optimal for a given level of risk aversion and constitutes a specific portfolio parameter. Institutional investors may have different degrees of risk tolerance for different components in their portfolio. For certain investment strategies, a reasonable measure of average risk aversion can be generated by examining the risk/return trade-off that investors typically choose. This is more difficult for a passive portfolio.

To see why, it is useful to think of a passive portfolio as being a bundle of two quite different consumables. The first is a set of factor exposures, be they small cap, growth, or non-U.S. equity. The second is a methodology by which the factor exposure values are generated. For the first kind of consumable, there are a multitude of choices for passive managers. For the second, there is really very little choice because passive managers rebalance in a very tight window around reconstitution events. Even if investors are uncomfortable with the performance implications of a given approach to passive management, they are extremely constrained in their ability to express this in portfolio choices. However, because the paths for an addition to the index at very high levels of risk aversion are reasonable approximations of what passive managers do in practice, the implications for the marginal risk/return trade-off can be explicitly calculated for an investor content with this approach. With $k = 1,000$, for example, the marginal return per unit of risk demanded by such an investor is approximately 45 basis points. One could take up the cudgels on behalf of such an estimate, given that the portfolio is supposed to be a passive investment. But they would have to be large cudgels indeed and skillfully wielded to carry the day. More reasonably, suppose that

some fraction of institutional investors actually has a far lower demand for return per unit of active risk. For such investors, a preferable policy for portfolio rebalancing would be one that is perhaps nearer in shape to, say, $k = 10$. This is interesting because it is not a shape currently on offer in typical passive management strategies.

4. Performance Implications

Table 1 summarizes the results of a simulation of the model derived in the Appendix. Because results for enhanced or active portfolios would be extremely dependent on the forecasting model used for excess return, the focus here is on passive portfolios that track a given equity index. Starting with the assumption that a manager knows which stocks will be added or subtracted from an index in advance of its reconstitution, the simulation generates results for the discrete switching approach to portfolio rebalancing, the current industry standard, and the smoothed trading approach. The relative performance of each approach is expressed in terms of turnover, trading costs, excess return, tracking error, and information ratio. The excess returns of each approach result from trading into and out of stocks that constitute the reconstitution event. These excess returns may result from more efficient trading around the reconstitution and also exposure to price movements generated by other investors trading once the list of additions to and deletions from the index becomes more widely known.

For each level of risk aversion, the smoothed approach results in a markedly higher excess return and also higher tracking error, which, for the avoidance of doubt, is always measured against the benchmark composition applying at that moment. At the highest level of risk aversion, excess return increases by approximately 70 basis points and the tracking error by roughly 20 basis points, for a marginal information ratio in excess of three. At somewhat lower levels of risk aversion, the improvement in excess return is smaller but so is the increase in tracking error between standard practice and optimal smoothing policies. These offsetting changes imply that at all levels of risk aversion considered here, a material improvement in risk-adjusted returns will result if the optimal policy is used.

Table 1: Simulated Performance with and without Reconstitution Smoothing

	k	% Turnover	Trading costs (bps)	Excess Return (bps)	σ (bps)	IR
Switch	1,000	40.16	10.28	-17.64	30.55	-0.58
	100	38.07	8.82	-13.11	32.83	-0.40
	10	34.65	7.56	-12.34	38.51	-0.32
	3	33.17	7.20	-18.32	43.75	-0.42
	2	32.59	7.08	-21.94	46.65	-0.47
Smoothed	1,000	49.73	11.65	51.00	52.29	0.98
	100	40.86	8.91	36.53	49.76	0.73
	10	34.66	7.36	19.85	51.37	0.39
	3	32.89	6.96	8.85	54.89	0.16
	2	32.19	6.82	3.63	57.18	0.06

The explicit transaction-cost component of returns is broken out separately. This helps bring home the point that the performance improvement is not solely the result of lower transaction costs, at least in the narrow sense of slippage and commissions, but is in much larger measure driven by the greater flexibility in trading and benchmark tracking permitted by the optimal policy. In other words, the salmon aren't better at swimming, they are just more canny about where they jump out of the water.

5. Conclusion

This paper advocates trading into positions by optimizing for excess return and tracking error outcomes as a more efficient way of managing around benchmark reconstitutions in the face of material transaction costs.

There are trade-offs, however. A performance improvement driven by lower transaction costs will presumably persist as long as the transaction-cost regime producing the results is maintained. The same cannot be said of the performance improvement here, since it depends on the actions of other market participants. Another health warning is that while past years have seen quite substantial reconstitution events, recent changes in methodology have reduced the likelihood of seeing similarly large events in the future (see Russell Investments [2008]). Nevertheless, the performance improvements that accrue to smoothed trading in advance of index reconstitutions are considerable, and they suggest that investment managers and plan sponsors should consider the approach outlined in this paper.

6. Appendix

Slippage is assumed to be the only transaction cost incurred² and depends on the square root³ of the ratio of trade size to average daily volume, such that slippage per share is given by:

$$s(\dot{w}(t), t) = a \left| \frac{\dot{w}(t)V(t)}{adv(t)p(t)} \right|^{1/2} \quad (1)$$

where $V(t)$ is the value of the portfolio, $p(t)$ is the price of the stock, $\dot{w}(t)$ is the change in weight of the stock in the portfolio, and $adv(t)$ is the stock's average daily share volume, in each case at time (t) . Total transaction costs as a proportion of portfolio value are then given by:

$$c(\dot{w}(t), t) = \alpha(t) |\dot{w}(t)|^{3/2} \quad (2)$$

where $\alpha(t) = a \left| \frac{V(t)}{adv(t)p(t)^3} \right|^{1/2}$. We assume that $\alpha(t) = \alpha$ and that the volatility of the stock is

likewise constant, $\sigma(t) = \sigma$. Utility at time t net of transaction costs is assumed to be:

$$u(w(t), \dot{w}(t), t) = -\alpha |\dot{w}(t)|^{3/2} - k(w(t) - b(t))^2 \sigma^2 \quad (3)$$

where k is risk aversion, and $w(t)$ is the weight of the stock in the portfolio and $b(t)$ is the stock's benchmark weight, both at time (t) . Over the interval 0 to T , where $0 < t_r < T$ is the date of the reconstitution, total utility is given by:

$$U(w, \dot{w}) = \int_0^T -\alpha |\dot{w}(t)|^{3/2} - k(w(t) - b(t))^2 \sigma^2 dt \quad (4)$$

To solve using optimal control theory, the control variable is set equal to the rate of the trading so that the maximization problem is:

$$\max_v J[w(t), v(t), t] = \int_0^T -\alpha |v(t)|^{3/2} - k(w(t) - b(t))^2 \sigma^2 dt \quad (5)$$

$$\dot{w}(t) = v(t) \quad (6)$$

$$w(0) = b(0) \quad (7)$$

$$w(T) = b(T) \quad (8)$$

There are two cases of interest: a step-function increase and a step-function decrease in the weight of a benchmark asset. Analytically, there is no meaningful difference between the two, and in the case of a stock entering the benchmark, the problem can usefully be structured as follows:

$$\begin{aligned} \max_{v, \hat{w}} J[w(t), v(t), t] &= \int_0^{t_r} -\alpha v(t)^{3/2} - k(w(t))^2 \sigma^2 dt + \\ &+ \int_{t_r}^T -\alpha v(t)^{3/2} - k(w(t) - b)^2 \sigma^2 dt \end{aligned} \quad (9)$$

$$\dot{w}(t) = v(t) \quad (10)$$

$$w(0) = 0 \quad (11)$$

$$w(t_r) = \hat{w} \quad (12)$$

$$w(T) = b \quad (13)$$

$$v(t) \geq 0 \quad (14)$$

These are two optimal control problems with an additional free variable \hat{w} that is a boundary value for both. The structure of each subproblem is essentially identical. For the first, the optimal path over 0 to t_r , the Hamiltonian and Lagrangian are:

$$H(w, v, t, \lambda) = -\alpha v(t)^{3/2} - k(w(t))^2 \sigma^2 + \lambda v \quad (15)$$

$$L(w, v, t, \lambda, \mu) = -\alpha v(t)^{3/2} - k(w(t))^2 \sigma^2 + \lambda v + \mu v \quad (16)$$

The first order conditions (see Caputo [2005]) that determine the optimal strategy over the second subproblem, the interval $0 \rightarrow t_r$, are:

$$\dot{\lambda}(t) = -H_w = 2kw(t)\sigma^2 \quad (17)$$

$$\dot{w} = H_{\lambda(t)} = v(t) \quad (18)$$

$$L_v = -(3/2)\alpha v(t)^{1/2} + \lambda(t) + \mu(t) = 0 \quad (19)$$

where $v(t) \geq 0, \mu(t) \geq 0, \mu(t)v(t) = 0$. It is reasonable to expect that for some $0 \rightarrow \hat{t}$ no trading occurs. From this second point onwards, $v > 0 \rightarrow \mu = 0$, which simplifies the solution. After some rearrangement of the preceding first-order conditions, the following ordinary differential equation is obtained for this subproblem:

$$\ddot{\lambda} = \gamma \lambda^2 \quad (20)$$

where $\gamma = \frac{8k\sigma^2}{9\alpha^2}$. This is a special case of the Emden-Fowler equation:

$$y_{xx}'' = Ax^n y^m \quad (21)$$

which in this case has a solution (see Polyanin and Zaitsev [2003]) that can be written in parametric form as:

$$t = gC_1^{-1} \tau \quad (22)$$

$$\lambda = hC_1^2 \wp \quad (23)$$

$$\gamma = \pm 6g^{-2}h^{-1} \quad (24)$$

where $\wp = \wp(\tau + C_2, 0, 1)$ is the Weierstrass elliptic function when taking the positive root in the implicit definition of τ (which is the root providing the solution required here) and where g and h are free parameters.

$$\tau = \int (\pm 4\wp^3 - 1)^{-1/2} d\wp - C_2 \quad (25)$$

Fixing $g = 1$ and rearranging after some manipulation of equations leads to:

$$\lambda(t) = \frac{27C_1^2\alpha^2}{4k\sigma^2} \wp[C_1t + C_2, 0, 1] \quad (26)$$

and using Equation (17) to:

$$w(t) = \frac{27C_1^3\alpha^2}{8k^2\sigma^4} \wp'[C_1t + C_2, 0, 1] \quad (27)$$

Given that $w(0) = 0$ and $w(t_r) = \hat{w}$, Equation (9) can be solved explicitly for $w(t)$. The structure of the solution is essentially identical for the interval, $t_r \rightarrow T$, the only change being the inclusion of a benchmark term and obviously different boundary values. Given numerical values for the constants, the optimal time path for $w(t)$ and the optimal value of \hat{w} can be obtained.

7. Notes

1. The Russell 3000[®] Index and Russell 2500[™] Index (the “Russell Indexes”) have been used for comparative purposes. Each of the Russell Indexes is a trademark/service mark of the Frank Russell Company. Russell[®] is a trademark of the Frank Russell Company. The Frank Russell Company is the source and copyright owner of the Russell Indexes and the return information relating to the Russell Indexes. No member of, or fund managed by, the D. E. Shaw group and no presentation in this document is sponsored or endorsed by, or affiliated or associated with, the Frank Russell Company. No member of the D. E. Shaw group sponsors or endorses the Frank Russell Company or the Russell Indexes.
2. We believe ignoring other costs is reasonable given the aim of finding a constraint or benchmark path and is a realistic approximation for most institutional portfolios where ticket charges and other fixed costs would be negligible.
3. See Torre (1997) for a detailed justification of this square-root relationship.

8. References

- [1] Caputo, Michael R. (2005). *Foundations of Dynamic Economic Analysis*. Cambridge: Cambridge University Press.
- [2] Polyanin, Andrei D., and Valentin F. Zaitsev (2003). *Handbook of Exact Solutions for Ordinary Differential Equations*, 2nd ed. Boca Raton: Chapman & Hall/CRC.
- [3] Russell Investments (2008). *Russell U.S. Equity Indexes: Construction and Methodology*.
- [4] Torre, Nicolo G. (1997). *BARRA Market Impact Model Handbook*. Berkeley: BARRA, Inc.

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